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On the Sufficiency of a Numerical Downstream Continuation

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1. INTRODUCTION

Customarily one does not impose n-th order boundary conditions on the solution of initial/boundary value problems whose characterizing partial differential equations are also n-th order. However, conjecture that such problems are not well-posed, or that a solution might not exist, is not always justified [1]. Perhaps a physically more natural example is provided by problems of computational fluid dynamics. Here boundary conditions which correctly should be applied at an infinite distance downstream from the region of interest are for computational convenience often applied at a finite location [2]. Results of numerical experimentation on viscous flows governed by the Navier–Stokes equations indicate that downstream continuation achieved by applying a second derivative boundary condition at a finite location often provides the least restrictive method of closing the flow [3].

In this investigation the mathematical impact of such bizarre boundary conditions on existence of solution is analyzed. Results from the theory of linear operators in Hilbert space are employed to establish the existence of solutions for the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \leqslant x \leqslant 1, \quad t > 0 \\ &u(0, t) = u_{xx}(1, t) = 0 \\ &u(x, 0) = f(x). \end{aligned} \tag{1}$$

An unconventional Hilbert space problem setting allows the insight that the question of existence is eqivalent to the question of completeness for the eigenfunctions of an essentially self adjoint linear differential operator, whose outcome is well-known [4].

2. PRELIMINARY CONSIDERATION

For convenience, we make the change of variable

$$u(x, t) = e^{x/2} v(x, t). \tag{2}$$

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Problem (1) then becomes

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{v}{4} &= \frac{\partial^2 v}{\partial x^2} \\ v(0, t) &= 0 \\ v_{xx}(1, t) + \frac{v(1, t)}{4} + v_x(1, t) &= 0 \\ v(x, 0) = g(x) &= e^{-x/2} f(x). \end{aligned} \tag{3}$$

The classical theory of Fourier Series [5] and separation of variables allows one to obtain a solution of (3), provided that eigenfunctions of the eigenvalue problem now indicated are complete[1] in the space of functions from which $g(x)$ is chosen: define an operator T by the relations

$$TF = \frac{-d^2F}{dx^2} + \frac{F}{4} = \lambda F \tag{4}$$

$$F(0) = 0, \qquad -F''(1) = \frac{F(1)}{4} + F'(1). \tag{5}$$

Here

$$(\ )' = \frac{d}{dx}(\ ), \qquad \lambda = \frac{1}{4} + \omega^2, \tag{6}$$

and ω is a positive root of

$$\tan \omega = \omega/(\omega^2 - \tfrac{1}{4}). \tag{7}$$

3. Hilbert Space Problem Setting

Let ψ_n, ψ_m be two eigenfunctions of the eigenvalue problem represented by equations (4–5). Employing (4–5) we see that the inner product

$$(\psi_n, \psi_m) = \int_0^1 \psi_n(x)\, \psi_m(x)\, dx \tag{8}$$

may be written as

$$\int_0^1 \psi_n \psi_m \, dx = -\psi_m(1)\, \psi_n(1). \tag{9}$$

[1] Here completeness means that an arbitrary function in the space can be expressed as a linear combination of the eigenfunctions.

This observation is the key to the completeness problem for the eigenfunctions of (4–5). Namely, if we consider a real Hilbert space topologized by

$$\|f\|_1 = \left[\int_0^1 f^2\,dx + f^2(1)\right]^{1/2} = (f,f)_1^{1/2}, \tag{10}$$

we immediately have orthogonal eigenfunctions!

This Hilbert space should contain at least all functions in the set

$$\begin{gathered} D_T = \{f(x): f(x) \text{ is square integrable;} \\ f(0) = 0, \text{ and } -f''(1) = f(1)/4 + f'(1); \\ \text{with } f, f' \text{ continuous and } f'' \text{ sectionally continuous on } \Omega\}. \end{gathered} \tag{11}$$

Since a Hilbert space is complete, and since D_T is not, we shall imbed D_T in a complete Hilbert space as follows.

Consider the transformation

$$Ag = -g(1)\left[e^{1/2(x-1)} - e^{-1/2(x+1)}\right] + e^{1/2x}\int_0^x e^{\alpha}\int_1^{\alpha} g(u)\,du\,d\alpha, \tag{12}$$

defined for square-integrable functions $g(x)$ which are at least piecewise continuous on Ω. Let D_A be the set of functions on which A is defined, and observe that every function in D_T is also in the domain D_A of A. We shall take as object Hilbert space the completion in the norm (10) of D_A, which shall be designated as $\tilde{L}_2(\Omega)$.

Lemma 1. *The operator A is a densely defined semibounded linear operator with inverse $A^{-1} = T$.*

Proof. By the definition of $\tilde{L}_2(\Omega)$, A us densely defined. Moreover, if $f(0) = 0$, then $f(x)$, by application of the Cauchy–Schwartz in equality, may be shown to satisfy

$$|f(x)|^2 \leqslant \int_0^1 f'(x)^2\,dx. \tag{13}$$

Since this implies

$$\int_0^1 f'(x)^2\,dx \geqslant f^2(1), \tag{14}$$

we may integrate by parts to show, for each $f(x)$ in D_T, that

$$(Tf,f)_1 = \int_0^1 \left[(f')^2 + \frac{f^2}{4}\right] dx + \frac{1}{2} f(1)^2 \geqslant \frac{1}{4}(f,f)_1 . \tag{15}$$

This implies $\|Tf\|_1 \geqslant \frac{1}{4}\|f\|_1$, for all f in D_T. Hence T possesses an inverse T^{-1} [6] which satisfies

$$(g, T^{-1}g)_1 \geqslant \tfrac{1}{4}(T^{-1}g, T^{-1}g)_1 \geqslant 0. \tag{16}$$

However, the solution of the equation

$$Tf = g$$

yields

$$f = T^{-1}g = Ag$$

where Ag is defined by (12). Hence $T^{-1} = A$. Equation (16) implies A is semibounded.

We characterize A as a bounded continuous operator in $\tilde{L}(\Omega)$, with dense domain. We infer results concerning the completeness of the eigenfunctions of T by considering the completeness question for the eigenfunctions of A.

4. Investigation of Completeness

Since A is densely defined, whenever the question

$$(Af, g)_1 = (f, g^*)_1 ,$$

with f in D_A and g in $\tilde{L}_2(\Omega)$, possesses a solution g^* in $\tilde{L}_2(\Omega)$, g^* is unique, and the operator A^* defined by

$$A^*g = g^*$$

is closed and has dense domain [7]. A^* is called the adjoint of A. Since A is invertible, A^* is invertible and

$$[A^*]^{-1} = [A^{-1}]^*;$$

hence, T^* exists [7].

Lemma 2. *The operator A is a symmetric operator.*

Proof. The operator A is symmetric provided

$$(Af, g)_1 = (f, Ag)_1 ,$$

for all f, g, in D_A. It is *self adjoint* if $D_A = D_A^*$, and $A = A_A^*$. We attack the symmetry problem for A by means of the symmetry problem for $T = A^{-1}$.

If we integrate by parts, then for U, V in D_T

$$(TU, V)_1 = (U, TV)_1 + [UV' - U'V + UV'' - U''V])|_0^1 . \tag{17}$$

But

$$U(0) = V(0) = 0, \qquad U(1)\,[V(1) + V\,(1)] = \frac{U(1)\,V(1)}{4},$$

and

$$V(1)\,[U'(1) + U''(1)] = \frac{U(1)\,V(1)}{4}.$$

Hence, the extreme right member of (17) vanishes and

$$(TU, V)_1 = (U, TV)_1 .$$

However, for each f, g in $R_T = D_A$, there exists U, V in D_T for which

$$U = T^{-1}f, \qquad V = T^{-1}g;$$

hence,

$$(f, Ag)_1 = (Af, g)_1 .$$

It follows that A is symmetric, in conjunction with T.

A cannot be self adjoint, since T is not *closed* (T is closed iff T^{-1} is closed). We can readly construct sequences $\{U_n\}$ in D_T which have no limit in D_T. However, for such sequences there exists U in $\tilde{L}_2(\Omega)$ and f in $\tilde{L}_2(\Omega)$, with $U = \lim_{n\to\infty} U_n$ and $f = \lim_{n\to\infty} TU_n$.

Moreover, the operator A admits closure; every densely defined semibounded symmetric operator admits a closed extension $\tilde{A}$ which is *self-adjoint* [6]. This is accomplished by extending D_A and D_T to include the limits U, f discussed above A-convergent sequences, and defining $TU = f$. Therefore, A is *essentially self-adjoint* [4].

THEOREM 1. *The eigenfunctions of the operator T are complete in $\tilde{L}_2(\Omega)$.*

Proof. Note that T and A possess the same eigenfunctions, and reciprocal real eigenvalues. Now, since A is essentially self-adjoint, the spectrum of A over D_A coincides with the spectrum of $\tilde{A}$ over $\bar{D}_A$ [4]. However, A has no continuous spectrum; and its limit spectrum is $\{0\}$. Thus, A has no eigenpackets; hence we know its eigenfunctions are complete in $\tilde{L}_2(\Omega)$ [4].

5. CONCLUDING REMARKS

Having established completeness of the eigenfunctions of (4–5), it is a fairly simple exercise to verify solution existence for problem (1), employing the methodology of [5]. Moreover, uniqueness of solution readily follows. If we assume that two distinct solutions U_1, U_2 exist, the difference function $H(x, t) = U_1 - U_2$ must vanish, as it is a solution of equation (3) with $g(x) = 0$. (Therefore, the Fourier coefficients in the eigenfunction expansion of H vanish.)

We have established that a unique solution of the problem characterized by equation (3) exists. Such interesting and illustrative examples allows one the feeling that in many circumstances the downstream continuations often employed in numerical fluid dynamics do indeed have a firm mathematical foundation.

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